The Annotated Applebaum

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0 Overview

[0.1] **Poisson random measures (p. xxiv):** Let $N(t)$ be a 1-dimensional ($d = 1$) Poisson process of intensity $\lambda > 0$ and jump size $h \neq 0$, the corresponding Lévy measure then is

$$\nu(dx) = \lambda\,\delta(x-h)\,dx$$

as can be seen from the remark that the (drift/Brownian free) *Lévy-Khinchin formula* then is

$$\eta(u) = \int_{y\neq 0} (e^{iuy}-1)\,\nu(dy) = \lambda(e^{iuh}-1)$$

which is the logarithmic characteristic of the said Poisson process $N(t)$ in the sense that

$$\mathbb{E}\left[e^{iuN(t)}\right] = e^{t\eta(u)} = e^{\lambda t(e^{iuh}-1)}$$

In this case, with the random times $T_n,\ n\in\mathbb{N}$, the jumps $\Delta X(s)$ of the Lévy process in the formula (0.3) are

$$\Delta X(s) = \begin{cases} h, & \text{for } s = T_k, \text{ for some } k,\\ 0, & \text{elsewhere}\end{cases}$$

and hence apparently

$$\sum_{0\leq s\leq t} \Delta X(s) = N(t)$$

In this case for the random measure

$$N(t,A) = \#\{0\leq s\leq t:\ \Delta X(s)\in A\}$$

we have

$$hN(t,A) = \begin{cases} N(t), & \text{for } h \in A, \\ 0, & \text{elsewhere} \end{cases}$$

namely we can write

$$N(t,dx) = \frac{N(t)}{h}\,\delta(x-h)\,dx$$

so that we finally find

$$\int_{x\neq 0} xN(t,dx) = \frac{N(t)}{h}\int_{x\neq 0} x\delta(x-h)\,dx = N(t) = \sum_{0\leq s\leq t}\Delta X(s)$$

as required at the page xxiv

[0.2] Diffusion equation (p. xxiv): The transition *pdf* $p(x,t|z,s)$ of the (1-dimensional) standard Brownian motion $B(t)$ satisfies the diffusion equation

$$\partial_t p(x,t|z,s) = \frac{1}{2}\,\partial_x^2 p(x,t|z,s)$$

with the initial condition $p(x,s^+|z,s) = \delta(x-z)$. Then, if $f(x)$ is a bounded continuous function, also the function

$$\begin{aligned} u(t,x) &= \int_{\mathbb{R}} p(x,t|z,0)f(z)\,dz \\ u(0,x) &= \int_{\mathbb{R}} p(x,0^+|z,0)f(z)\,dz = \int_{\mathbb{R}} \delta(x-z)f(z)\,dz = f(x) \end{aligned}$$

is a solution of the diffusion equation:

$$\begin{aligned} \partial_t u(t,x) &= \int_{\mathbb{R}} \partial_t p(x,t|z,0)f(z)\,dz = \frac{1}{2}\int_{\mathbb{R}} \partial_x^2 p(x,t|z,0)f(z)\,dz \\ &= \frac{1}{2}\,\partial_x^2 \int_{\mathbb{R}} p(x,t|z,0)f(z)\,dz = \frac{1}{2}\,\partial_x^2 u(t,x) \end{aligned}$$

with the initial condition $u(0,x) = f(x)$. On the other hand for our standard Brownian motion

$$p(x,t|z,s) = p_{t-s}(x-z)$$

where $p_t(x) = p(x,t|0,0)$ is the symmetric *pdf* of the Gaussian law $\mathfrak{N}(0,t)$ referred to in the text. As a consequence, by taking $y = z - x$ and by using the symmetry of $p_t(x)$, we have

$$\begin{aligned} u(t,x) &= \int_{\mathbb{R}} p(x,t|z,0)f(z)\,dz = \int_{\mathbb{R}} f(z)p_t(x-z)\,dz \\ &= \int_{\mathbb{R}} f(x+y)p_t(y)\,dy = \mathbb{E}\left[f(x+B(t))\right] \end{aligned}$$

as suggested at the page xxiv

1 Lévy processes

1.1 Review of measure and probability

[1.1.1] Conditioning (p. 10): Since $\mathbb{E}[X|\mathcal{G}]$ is the Radon-Nikodým derivative of Q_X with respect to P, from the Radon-Nikodým Theorem we have that

$$Q_X(A) = \int_A \mathbb{E}[X|\mathcal{G}](\omega)\, P(d\omega)$$

for every $A \in \mathcal{G} \subseteq \mathcal{F}$ (but not possibly for every $A \in \mathcal{F}$). From the very definition of $Q_X(A)$ we then deduce that

$$\int_A X(\omega)\, P(d\omega) = \int_A \mathbb{E}[X|\mathcal{G}](\omega)\, P(d\omega) \qquad \forall A \in \mathcal{G}$$

Remark that this *does not imply* that $X = \mathbb{E}[X|\mathcal{G}]$, P-a.s. because the previous relations does not hold for every $A \in \mathcal{F}$, but only for every $A \in \mathcal{G} \subseteq \mathcal{F}$. The proposed definition is indeed a generalization of a more elementary one: for two *rv*'s X, Y we first define in an heuristic way the Borel function

$$m_X(y) \equiv \mathbb{E}[X|Y = y] \qquad y \in \mathbb{R}$$

and then the conditional expectation of X with respect to the *rv* Y as

$$\mathbb{E}[X|Y](\omega) \equiv m_X(Y(\omega))$$

As a matter of fact, if $\mathcal{G}_Y$ is the σ-algebra generated by Y, we finally find that

$$\mathbb{E}[X|Y] = \mathbb{E}[X|\mathcal{G}_Y]$$

with the right hand side defined as in the Applebaum book. However, in this more general (and more rigorous) definition, the sub-σ-algebra $\mathcal{G} \subseteq \mathcal{F}$ is not in general required to be generated by a *rv* whatsoever

[1.1.2] Ky Fan metric (p. 15): Apparently $P\{|X - Y| > \epsilon\}$ is a non-increasing function of $\epsilon > 0$ with

$$\lim_{\epsilon \downarrow 0} P\{|X - Y| > \epsilon\} = 1$$

so that (as the Figure 1 shows) somewhere in $[0, 1]$ its value will fall below ϵ itself: the smallest ϵ realizing this condition is the Ky Fan distance $d(X, Y)$

1.2 Infinite divisibility

[1.2.1] Lévy measure (p. 29): Put $d = 1$ for simplicity. When the Lévy measure admits a *pdf* $\ell(y)$ then the condition (1.10) becomes

$$\int_{\mathbb{R}-\{0\}} (y^2 \wedge 1)\ell(y)\, dy < +\infty$$

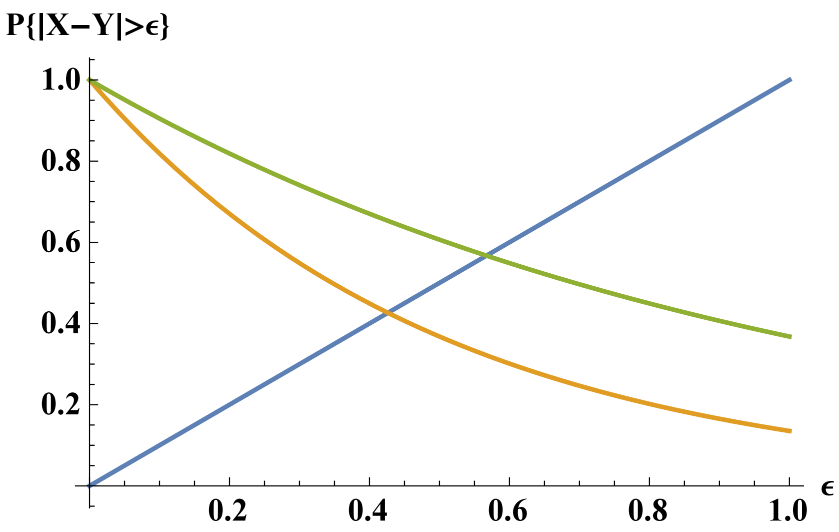

Figure 1: Qualitative plot for the Ky Fan metric

This means that $\ell(y)$ can be singular in $y = 0$ at most as $y^{-2-\alpha}$ with $0 \leq \alpha < 1$. For $y \to \infty$ on the other hand $\ell(y)$ must be infinitesimal at least as $y^{-1-\alpha}$ in order to be integrable as required in (1.10)

[1.2.2] **The Lévy-Khintchine formuula (p. 29):** Always with $d = 1$, remark that $1 + iuy$ are the first two terms of the Taylor expansion of e^{iuy}. The term $iuy\chi_{\hat{B}}(y)$, moreover, is non zero only in $\hat{B} = B_1(0) = (-1, 1)$. This means on the one hand that $e^{iuy} - 1 - iuy\chi_{\hat{B}}(y)$ is $O(y^2)$ for $y \to 0$ so that the integral in (1.12) certainly converges around $y = 0$ because of (1.10). For $y \to \infty$, on the other hand, (1.10) guarantees the convergence for every bounded function as $e^{iuy} - 1$ is: remark moreover that we could not keep the term iuy in the asymptotic region because it is not bounded, and this is the motivation for introducing the cutoff $\chi_{\hat{B}}(y)$

When in particular the Lévy measure ν is a probability the singularity of its *pdf* $\ell(y)$ in $y = 0$ can be at most $y^{-\alpha}$ with $0 \leq \alpha < 1$, and hence in the integral in the Lévy-Khintchine formula: (1) we are no longer obliged to pull the point $\{0\}$ out from the integration domain $\mathbb{R}$, and (2) the term $iuy\chi_{\hat{B}}(y)$ is no longer needed for the convergence in $y = 0$. In this case we can then dispose of this last term by shifting the integral

$$-\int_{\mathbb{R}} iuy\chi_{\hat{B}}(y)\,\nu(dy) = -iu\int_{-1}^{1} y\,\nu(dy)$$

in the first term by modifying the coefficient b into

$$b' = b - \int_{-1}^{1} y\,\nu(dy)$$

and rephrasing the Lévy-Khintchine formula as (for $d = 1$ the matrix A is a number)

$$\phi_\mu(u) = \exp\left\{ib'u + \frac{A}{2}u^2 + \int_{\mathbb{R}} (e^{iuy} - 1)\,\nu(dy)\right\}$$

which (with a suitable reinterpretation of the symbols) apparently coincides with the formula (1.9) representing the characteristic function of a sum of independent Gaussian and compound Poisson *rv*'s

[1.2.3] A proof (p. 30): Remark that

$$U(n)^c \xrightarrow{n} \mathbb{R}^d - \{0\}$$

and hence also

$$\int_{U(n)^c\cap\hat{B}} \dots\,\nu(dy) = \int_{U(n)^c} \dots\,\chi_{\hat{B}}(y)\,\nu(dy) \xrightarrow{n} \int_{\mathbb{R}^d-\{0\}} \dots\,\chi_{\hat{B}}(y)\,\nu(dy)$$

so that, as stated in the text,

$$\phi_\mu(u) = \lim_n \phi_n(u)$$

To prove then that $\psi_\mu(u)$ is continuous in $u = 0$ we first remark that $\psi_\mu(0) = 0$ because

$$\left[e^{i(u,y)} - 1 - i(u,y)\right]_{u=0} = 0$$

and then we check that also

$$\lim_{u\to 0} \psi_\mu(u) = 0$$

following the proof in the texi

Finally the infinite divisibility of the right hand side of (1.12) can be established either by remarking that $\phi_\mu^{1/n}(u)$ has the same form as (1.12) with $b/n, A/n$ and a Lévy measure $\frac{1}{n}\nu$, or by recalling the Exercise 1.2.12 (2) at p. 28 and the previous remark that the ϕ_μ is indeed the limit of a sequence of infinitely divisible characteristic functions ϕ_n

[1.2.4] A formula (p. 41): From Appendix 1.7 at p. 80 we know that

$$\int_0^\infty \frac{1-e^{-ux}}{x^{1+\alpha}}\,dx = \frac{\Gamma(1-\alpha)}{\alpha}\,u^\alpha \qquad \begin{cases} 0 \le u \\ 0 < \alpha < 1 \end{cases}$$

and hence for

$$u = 1 + \frac{|p|^2}{\lambda} \qquad \alpha = \frac{1}{2}$$

we have

$$\int_0^\infty \left(1 - e^{-\left[1+\frac{|p|^2}{\lambda}\right]x}\right)\frac{dx}{x^{3/2}} = 2\,\Gamma\left({}^1\!/_2\right)\sqrt{1+\frac{|p|^2}{\lambda}} = 2\sqrt{\pi}\sqrt{1+\frac{|p|^2}{\lambda}}$$

giving eventually the result at p. 41

[1.2.5] A relativistic Lévy measure (p. 42): To check that the proposed

$$g_\nu(x) = 2\left(\frac{2\pi|x|}{mc^2}\right)^{-\frac{d+1}{2}} K_{\frac{d+1}{2}}(mc^2|x|)$$

is the density of the Lévy measure ν of the triplet $(0, 0, \nu)$ for the relativistic characteristic function

$$\phi_{m,c}(p) = e^{-E_{m,c}(p)} \qquad - E_{m,c}(p) = mc^2\left(1 - \sqrt{1 + \frac{|p|^2}{m^2c^2}}\right)$$

we must show that, with $u = cp$, the Lévy-Khintchine formula

$$\int_{\mathbb{R}^d-\{0\}} \left(e^{ic(p,y)} - 1 - ic(p,y)\chi_{\hat{B}}(y)\right) g_\nu(y)\,dy = mc^2\left(1 - \sqrt{1 + \frac{|p|^2}{m^2c^2}}\right)$$

is satisfied. First of all let us remark that, since $g_\nu(y)$ is spherically symmetric, it is

$$\begin{aligned} -ic\int_{\mathbb{R}^d-\{0\}} (p,y)\chi_{\hat{B}}(y)g_\nu(y)\,dy &= 0 \\ \int_{\mathbb{R}^d-\{0\}} \left(e^{ic(p,y)} - 1\right) g_\nu(y)\,dy &= \int_{\mathbb{R}^d-\{0\}} \left[\cos\left(c(p,y)\right) - 1\right] g_\nu(y)\,dy \end{aligned}$$

and then that – by taking $d = 1$ for simplicity – again by symmetry we are reduced to evaluate

$$\begin{aligned} \int_{\mathbb{R}-\{0\}} \left[\cos\left(c(p,y)\right) - 1\right] g_\nu(y)\,dy &= \frac{mc^2}{\pi}\int_{y\neq0} \frac{\cos(cpy) - 1}{|y|} K_1(mc^2|y|)\,dy \\ &= \frac{2mc^2}{\pi}\int_0^\infty \frac{\cos(cpy) - 1}{y} K_1(mc^2y)\,dy \end{aligned}$$

Now, by taking $x = mc^2y$ and $\lambda = {}^p/_{mc} > 0$, we have

$$\int_{\mathbb{R}-\{0\}} \left[\cos\left(c(p,y)\right) - 1\right] g_\nu(y)\,dy = \frac{2mc^2}{\pi}\int_0^\infty \frac{\cos(\lambda x) - 1}{x} K_1(x)\,dx$$

and the result follows from the fact that

$$\int_0^\infty \frac{\cos(\lambda x) - 1}{x} K_1(x)\,dx = \frac{\pi}{2}\left(1 - \sqrt{1 + \lambda^2}\right)$$

as can be seen, for instance, with *Mathematica*

1.3 Lévy processes

[1.3.1] Jumps in Poisson processes (p. 50): The idea is that if two independent Poisson processes $N_1(t)$ and $N_2(t)$ could jump at the same time, the sum Poisson process $N(t) = N_1(t) + N_2(t)$ would be allowed to make length-2 jumps at these times, instead of only length-1 jumps as for every Poisson process. However the proof looks not altogether convincing because the invoked Exercise 1.3.6 states only that the sum of two independent Lévy processes again is a Lévy process: this means that the sum of two independent Poisson process is indeed a Lévy process, but does not seem to imply that it must also be a simple Poisson process

[1.3.2] The Laplace exponent (p. 52): The characterstic function of our subordinator $T(t)$ (which is a particular Lévy process) is

$$\phi_{T(t)}(u) = \mathbb{E}\left[e^{iuT(t)}\right] = e^{t\eta(u)}$$

where the Lévy symbol $\eta(u)$ of $T(1)$ has the form (1.23) in the text. Now, performing the replacement $u \to iu$ ($u > 0$) by analytical continuation, we get

$$\phi_{T(t)}(iu) = e^{t\eta(iu)} = e^{-t\psi(u)}$$

where we have defined the Laplace exponent $\psi(u)$ which from (1.23) is

$$\psi(u) = -\eta(iu) = bu - \int_0^\infty \left(e^{-uy} - 1\right)\lambda(dy)$$

as stated in (1.24)

[1.3.3] α-stable subordinators (p. 53): By taking $b = 0$ and

$$\lambda(dx) = \frac{\alpha}{\Gamma(1-\alpha)}\,\frac{dx}{x^{1+\alpha}}\chi_{(0.+\infty)}(x) \qquad\qquad 0 < \alpha < 1$$

and by using the mentioned integral representation of u^α, the formula (1.24) with $u \geq 0$ becomes

$$\psi(u) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \left(1 - e^{-ux}\right)\frac{dx}{x^{1+\alpha}} = u^\alpha$$

which by Theorem 1.3.15 is the Laplace exponent of a subordinator whose Lévy symbol (1.23) is

$$\eta(u) = -\psi(-iu) = -(-iu)^\alpha = -|u|^\alpha e^{-i\,\mathrm{sgn}(u)\frac{\alpha\pi}{2}}$$

On the other hand from Theorem 1.2.21 we know that for $0 < \alpha < 1$ the characteristic exponent of an α-stable law with $\mu = 0,\ \beta = 1$ and $\sigma^\alpha = \cos\frac{\alpha\pi}{2}$ is

$$-|u|^\alpha\sigma^\alpha\left(1 - i\,\mathrm{sgn}(u)\tan\frac{\alpha\pi}{2}\right) = -|u|^\alpha e^{-i\,\mathrm{sgn}(u)\frac{\alpha\pi}{2}}$$

so that our subordinators turn out to be particular non-Gaussian α-stable processes, the Gaussian case ($\alpha = 2$) being of course excluded

[1.3.4] The Lévy subordinator (p. 53): If in the previous remark [1.3.3] we take $\alpha = {}^1/_2$, from $\sigma^\alpha = \cos\frac{\alpha\pi}{2}$ we have

$$\sqrt{\sigma} = \cos\frac{\pi}{4} \qquad \text{namely} \qquad \sigma = \cos^2\frac{\pi}{4} = \frac{1}{2}$$

and hence (being $\tan\frac{\pi}{4} = 1$) the corresponding Lévy symbol becomes

$$\eta(u) = -\sqrt{\frac{|u|}{2}}\,\big(1 - i\,\mathrm{sgn}(u)\big)$$

As a consequence the $\frac{1}{2}$-stable (Lévy) subordinator $T(t)$ has the characteristic function

$$\phi_{T(t)}(u) = e^{t\eta(u)} = e^{-\sqrt{\frac{t^2}{2}|u|}\,\big(1-i\,\mathrm{sgn}(u)\big)}$$

We must remember now from the Theorem 1.2.21, and the subsequent particular cases at p. 36, that if T is a $\frac{1}{2}$-stable (Lévy) *rv* (with $\mu = 0$ and $\beta = 1$) then for some $c > 0$ its characteristic function and *pdf* respectively are

$$\begin{aligned}
\phi_T(u) &= e^{-\sqrt{c|u|}\,\big(1-i\,\mathrm{sgn}(u)\tan\frac{\pi}{4}\big)} = e^{-\sqrt{c|u|}\,\big(1-i\,\mathrm{sgn}(u)\big)} \\
f_T(s) &= \sqrt{\frac{c}{2\pi}}\,s^{-3/2}\,e^{-\frac{c}{2s}}
\end{aligned}$$

where we adopted the symbol c instead of σ in order to avoid confusion. By comparing the characteristic functions we immediately see that our Lévy subordinator is indeed a $\frac{1}{2}$-stable process provided that we take $c = {}^{t^2}/_2$, so that the corresponding *pdf* coincides with that suggested at p. 53

[1.3.5] Inverse Gaussian distribution (p. 54): The *pdf* of the Inverse Gaussian law $IG(\delta, \gamma)$ of $T(1)$ is

$$f(s; \delta, \gamma) = \frac{\delta\, e^{\delta\gamma}}{\sqrt{2\pi}}\,\frac{e^{-\frac{\delta^2}{2s} - \frac{\gamma^2 s}{2}}}{s^{3/2}}$$

A few examples of this *pdf* are displayed in the Figure 2 for several possible values of the pair (δ, γ)

[1.3.6] Bernstein functions (p. 55): From the given definitions it can be easily deduced that $f(x)$ is Bernstein when its first derivative is completely monotone. When in fact $f(x)$ is Bernstein we have $(-1)^n f^{(n)}(x) \leq 0$. Put now $g(x) = f'(x)$ so that $(-1)^n g^{(n-1)}(x) \leq 0$: it is apparent then that $(-1)^{n-1} g^{(n-1)}(x) \geq 0$, namely that $g(x)$ is completely monotone

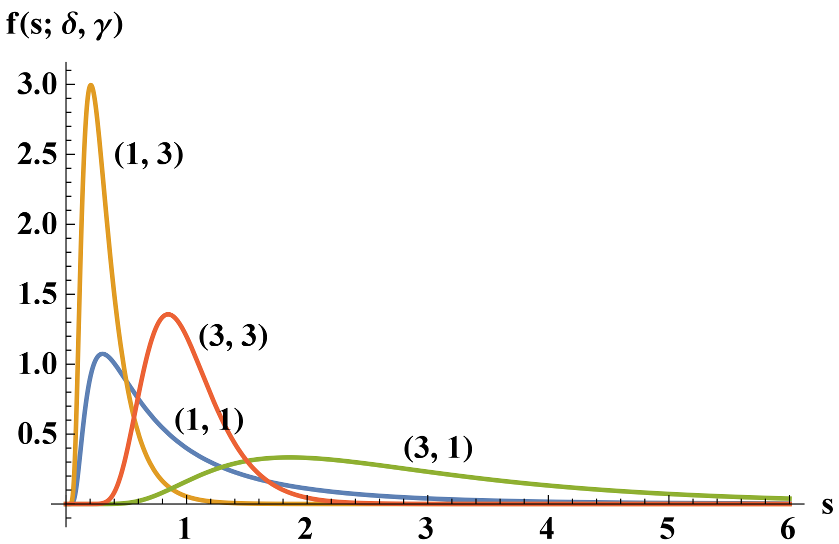

Figure 2: Examples of the Inverse Gaussian *pdf*

It can be seen by direct calculation that decreasing exponentials e^{-ax} and decreasing powers x^{-a} for $a > 0$ are completely monotone, that logarithms $\log(ax)$ are Bernstein, while powers, polynomials and increasing exponentials e^{ax} are neither Bernstein, nor completely monotone

[1.3.7] **Stationary increments (p. 57):** More precisely the chain of equations (taking into account the fact that $X(t), T(t)$ are independent Lévy processes) appears to be

$$\begin{aligned}
P\{Z(t_2) - Z(t_1) \in A\} &= P\{X\big(T(t_2)\big) - X\big(T(t_1)\big) \in A\} \\
&= \int_0^\infty \int_0^\infty P\{X\big(T(t_2)\big) - X\big(T(t_1)\big) \in A \,\big|\, T(t_1) = s_1, T(t_2) = s_2\} \; p_{t_1,t_2}(ds_1, ds_2) \\
&= \int_0^\infty \int_0^\infty P\{X(s_2) - X(s_1) \in A\} \; p_{t_1,t_2}(ds_1, ds_2) \\
&= \int_0^\infty \int_0^\infty P\{X(s_2 - s_1) \in A\} \; p_{t_1,t_2}(ds_1, ds_2) \\
&= \int_0^\infty \int_0^\infty P\{X\big(T(t_2) - T(t_1)\big) \in A \,\big|\, T(t_1) = s_1, T(t_2) = s_2\} \; p_{t_1,t_2}(ds_1, ds_2) \\
&= P\{X\big(T(t_2) - T(t_1)\big) \in A\} = P\{X\big(T(t_2 - t_1)\big) \in A\} = P\{Z(t_2 - t_1) \in A\}
\end{aligned}$$

[1.3.8] **Independent increments (p. 57):** According to Kac's Theorem 1.1.16, p. 18, we are reduced to prove for $d = 1$, $0 \le t_1 < t_2 < t_3$, and arbitrary y_1, y_2 that

$$\mathbb{E}\left[e^{iy_1\left(Z(t_2)-Z(t_1)\right)} e^{iy_2\left(Z(t_3)-Z(t_2)\right)}\right] = \mathbb{E}\left[e^{iy_1\left(Z(t_2)-Z(t_1)\right)}\right] \mathbb{E}\left[e^{iy_2\left(Z(t_3)-Z(t_2)\right)}\right]$$

To this end we first introduce the functions (conditional expectations)

$$\begin{aligned} h_y(s) &= \mathbb{E}\left[e^{iyZ(t)} \,\middle|\, T(t)=s\right] \\ f_{y_1,y_2}(u_1,u_2,u_3) &= \\ &= \mathbb{E}\left[e^{iy_1\left(Z(t_2)-Z(t_1)\right)}e^{iy_2\left(Z(t_3)-Z(t_2)\right)} \,\middle|\, T(t_1)=u_1, T(t_2)=u_2, T(t_3)=u_3\right] \end{aligned}$$

with the understanding that

$$\begin{aligned} h_y(T(t)) &= \mathbb{E}\left[e^{iyZ(t)} \,\middle|\, T(t)\right] \\ f_{y_1,y_2}(T(t_1),T(t_2),T(t_3)) &= \mathbb{E}\left[e^{iy_1\left(Z(t_2)-Z(t_1)\right)}e^{iy_2\left(Z(t_3)-Z(t_2)\right)} \,\middle|\, T(t_1),T(t_2),T(t_3)\right] \end{aligned}$$

and then we prove that

$$h_y(s) = \mathbb{E}\left[e^{iyX(s)}\right] \qquad\qquad f_{y_1,y_2}(u_1,u_2,u_3) = h_{y_1}(u_2-u_1)h_{y_2}(u_3-u_2)$$

From the fact that $Z(t)$ has stationary increments, that $0 \le u_1 < u_2 < u_3$ and that $X(t), T(t)$ are independent Lévy processes, we indeed have

$$\begin{aligned} h_y(s) &= \mathbb{E}\left[e^{iyZ(t)} \,\middle|\, T(t)=s\right] = \mathbb{E}\left[e^{iyX\left(T(t)\right)} \,\middle|\, T(t)=s\right] = \mathbb{E}\left[e^{iyX(s)}\right] \\ f_{y_1,y_2}(u_1,u_2,u_3) &= \\ &= \mathbb{E}\left[e^{iy_1\left(Z(t_2)-Z(t_1)\right)}e^{iy_2\left(Z(t_3)-Z(t_2)\right)} \,\middle|\, T(t_1)=u_1, T(t_2)=u_2, T(t_3)=u_3\right] \\ &= \mathbb{E}\left[e^{iy_1\left(X(T(t_2))-X(T(t_1))\right)}e^{iy_2\left(X(T(t_3))-X(T(t_2))\right)} \,\middle|\, T(t_1)=u_1, T(t_2)=u_2, T(t_3)=u_3\right] \\ &= \mathbb{E}\left[e^{iy_1\left(X(u_2)-X(u_1)\right)}e^{iy_2\left(X(u_3)-X(u_2)\right)}\right] \\ &= \mathbb{E}\left[e^{iy_1\left(X(u_2)-X(u_1)\right)}\right]\mathbb{E}\left[e^{iy_2\left(X(u_3)-X(u_2)\right)}\right] \\ &= \mathbb{E}\left[e^{iy_1X(u_2-u_1)}\right]\mathbb{E}\left[e^{iy_2X(u_3-u_2)}\right] = h_{y_1}(u_2-u_1)h_{y_2}(u_3-u_2) \end{aligned}$$

And finally from the usual properties of the conditioning we get the required result

$$\begin{aligned} \mathbb{E}\left[e^{iy_1\left(Z(t_2)-Z(t_1)\right)}e^{iy_2\left(Z(t_3)-Z(t_2)\right)}\right] &= \mathbb{E}\left[f_{y_1,y_2}\big(T(t_1),T(t_3),T(t_4)\big)\right] \\ &= \mathbb{E}\left[h_{y_1}(T(t_2)-T(t_1))h_{y_2}(T(t_3)-T(t_2))\right] \\ &= \mathbb{E}\left[h_{y_1}(T(t_2)-T(t_1))\right]\mathbb{E}\left[h_{y_2}(T(t_3)-T(t_2))\right] \\ &= \mathbb{E}\left[h_{y_1}(T(t_2-t_1))\right]\mathbb{E}\left[h_{y_2}(T(t_3-t_2))\right] \\ &= \mathbb{E}\left[e^{iy_1Z(t_2-t_1)}\right]\mathbb{E}\left[e^{iy_2Z(t_3-t_2)}\right] \\ &= \mathbb{E}\left[e^{iy_1Z(t_2)-Z(t_1)}\right]\mathbb{E}\left[e^{iy_2Z(t_3)-Z(t_2)}\right] \end{aligned}$$

[1.3.9] Subordinator symbol (p. 58): The first member of the chain of equalities apparently must be ($d = 1$)

$$e^{t\eta_Z(u)} = \mathbb{E}\left[e^{iuZ(t)}\right] = \mathbb{E}\left[e^{iuX\left(T(t)\right)}\right]$$

The result then follows without further difficulties

[1.3.10] Normal *IG* process (p. 60): Take first an *IG* subordinator

$$T(t) = \inf\{s > 0 \,:\, C_1(s) = \delta t\} \qquad\qquad \delta > 0$$

where $C_1(t) = B_1(t) + \gamma t$ where $\gamma > 0$ and $B_1(t)$ is a Brownian motion. Take now a second, independent Brownian motion $B(t)$, add a drift $C(t) = B(t) + \beta t$ with $\beta \in \mathbb{R}$ and define the *Normal IG process* as

$$Z(t) = C(T(t)) + \mu t = B(T(t)) + \beta T(t) + \mu t \qquad\qquad \mu \in \mathbb{R}$$

The basic ingredients are then two independent Brownian motions $B(t)$ and $B_1(t)$ and four real parameters: $\gamma, \delta, \beta, \mu$, but it is customary in the literature to replace γ with another real parameter α such that $\gamma = \sqrt{\alpha^2 - \beta^2}$ with the understanding that $\alpha^2 > \beta^2$. As a consequence every Normal *IG* process is characterized by the four real parameters $\alpha, \beta, \delta, \mu$ with $\delta > 0$ and $\alpha^2 > \beta^2$

[1.3.11] Subordinated Lévy symbol (p. 62): If $Z(t) = X(T(t))$ is a subordinated process, from the Proposition 1.3.27 we know that its Lévy symbol is

$$\eta_Z(u) = -\psi_T\big(-\eta_X(u)\big)$$

where ψ_T is the Laplace exponent of $T(t)$, and η_X is the Lévy symbol of $X(t)$. On the other hand from (1.24) we know that the Laplace exponent of a subordinator takes the form

$$\psi_T(u) = -\eta_T(iu) = b\,u + \int_0^\infty \left(1 - e^{-uy}\right) \lambda(dy)$$

so that

$$\eta_Z(u) = b\,\eta_X(u) + \int_0^\infty \left(e^{s\,\eta_X(u)} - 1\right) \lambda(ds)$$

and the result easily follows

References

[1] *D. Applebaum*: Lévy processes and Stochastic Calculus (Cambridge U.P., Cambridge, 2009).